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Resolvable gregarious cycle decompositions of complete equipartite graphs

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Abstract

The complete multipartite graph $K_{n(m)}$ with n parts of size m is shown to have a decomposition into n -cycles in such a way that each cycle meets each part of $K_{n(m)}$; that is, each cycle is said to be *gregarious*. Furthermore, gregarious decompositions are given which are also resolvable.

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1. Introduction

Edge-disjoint decompositions of various graphs into cycles have been considered by many authors. Necessary and sufficient conditions for a complete graph of odd order, or for a complete graph of even order minus a one-factor, to have a decomposition into cycles of some fixed length are now known; see [1,6], and references therein. Moreover, resolvable cycle decompositions of complete multipartite graphs have been considered by various authors; see for instance [5] and references therein. However, the requirement that each cycle in a decomposition of a complete multipartite graph has all its vertices in different parts of the graph is a new one. It was introduced in [2] by the first two authors, who gave necessary and sufficient conditions for a decomposition of any complete tripartite graph with parts of possibly different sizes into *gregarious* 4-cycles. This requirement meant that every 4-cycle had at least one vertex in each part of the tripartite graph.

In this paper such gregarious decompositions are considered further. We take a complete equipartite graph $K_{n(m)}$, with n parts of size m , and give an edge-disjoint decomposition into n -cycles in such a way that each n -cycle has one vertex in each of the partite sets; that is, each n -cycle is *gregarious*. We also present such gregarious decompositions which are *resolvable*; that is, the gregarious cycles partition into sets of m n -cycles which precisely cover all mn vertices.

The number of edges in $K_{n(m)}$ is $m^2 \binom{n}{2}$, and for an edge-disjoint decomposition the degree $m(n-1)$ of each vertex must be even. So if the size of each part, m , is odd, then necessarily the number of parts n is also odd.

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2. Results: number of parts n is even

In this case, with the number of parts n of $K_{n(m)}$ being even, the size m of each part is necessarily even as well.

We concentrate on parts of smallest possible size, $m=2$. The case with parts of size $2m$ arises easily from the case with parts of size 2, by replacing each part $\{x, x'\}$ by the part $\{x_i, x'_i \mid 1 \leq i \leq m\}$, and each (generic) cycle $(1, 2, 3, \dots, n)$ (where some entries may be primed) by the m^2 cycles $(1_i, 2_j, 3_i, \dots, n_j)$, $1 \leq i, j \leq m$.

Note that a gregarious n -cycle decomposition of $K_{n(2)}$ may also be regarded as an n -cycle decomposition of $K_{2n} - F$, where F is a one-factor of K_{2n} (corresponding to the pairs of edges forming the n parts of size 2 in $K_{n(2)}$), which is *orthogonal* to F . That is, each n -cycle meets each edge of F in at most one vertex—in fact in precisely one vertex, since there are n edges in F and n vertices in each cycle.

Lemma 2.1. *For all even n , there exists a gregarious n -cycle system of $K_{n(2)}$ which is also resolvable.*

Proof. We deal with two cases, according as n is 0 or 2 (mod 4).

First let $n = 4k$, and let $K_{n(2)}$ have parts $\{x, x'\}$ of size 2 where $x \in \{\infty\} \cup \mathbb{Z}_{4k-1}$. The required number of n -cycles in $K_{n(2)}$ is $4 \binom{n}{2} / n = 2(n-1) = 2(4k-1)$. In the case $k=1$, take $(\infty, 0, 1', 2')$, $(\infty', 0', 1, 2)$; these two starter cycles (mod 3) form a resolution class. For $k > 1$ we give two starter cycles modulo $4k-1$:

$$\begin{aligned} &(\infty, 0, 1', 4k-2, 2', 4k-3, 3', 4k-4, 4', \dots, (k-1)', 3k, k', (3k-1)', (k+1)', (3k-2)', \dots, \\ &(2k-1)', (2k)'), \\ &(\infty', 0', 1, (4k-2)', 2, (4k-3)', 3, (4k-4)', 4, \dots, (k-1), (3k)', k, 3k-1, k+1, 3k-2, \dots, \\ &2k-1, 2k). \end{aligned}$$

Note that these starter cycles form a resolution class. It is easily checked that the first starter covers mixed differences $1, 2, \dots, 2k-1$ and all pure primed differences. The second starter likewise covers the remaining $2k-1$ mixed differences, and all the $2k-1$ pure (unprimed) differences. The mixed difference 0 is, of course, used in the parts of size 2. Thus these starters yield a resolvable gregarious $4k$ -cycle system of $K_{2n} - F$ or $K_{n(2)}$.

Secondly, let $n = 4k+2$, and let $K_{n(2)}$ have parts $\{x, x'\}$ of size 2 where $x \in \{\infty\} \cup \mathbb{Z}_{4k+1}$. Again the number of cycles is $2(n-1) = 2(4k+1)$, and we take two starter cycles modulo $(4k+1)$, $k > 1$. (The case $n=6$, when $k=1$, is given in Example 2.3 below.)

$$\begin{aligned} &(\infty, 2k+1, 0, 1', 4k, 2', 4k-1, 3', 4k-2, 4', \dots, (2k-1)', 2k+2, (2k)'), (\infty', k+1, k, k+2, k-1, \\ &k+3, k-2, \dots, 2k-1, 2, 2k, 1, 0', (2k+1)'(4k)', (2k+2)', (4k-1)', (2k+3)', \dots, \\ &(3k+2)', (3k)', (3k+1)'). \end{aligned}$$

These two starter cycles together form a resolution class. The first starter covers mixed differences $1, 2, \dots, 4k-1$ together with the pure (unprimed) difference $2k$. The second starter covers the single mixed difference $4k$, together with all the pure primed differences $1, 2, \dots, 2k$, and pure unprimed differences $1, 2, \dots, 2k-1$. Thus these two starter cycles give a resolvable gregarious n -cycle system of $K_{2n} - F = K_{n(2)}$ when n is 2 modulo 4.

This completes the lemma. \square

Corollary 2.2. *When n is even, there is a gregarious n -cycle system of $K_{n(m)}$ which is also resolvable.*

Proof. From each resolution class in the resolvable gregarious n -cycle decomposition of $K_{n(2)}$ in the above lemma, we blow up each point by $m/2$ (so that parts of size 2 become parts of size m with m even), and we replace any cycle of the generic form $(1, 2, \dots, n)$ by the $m^2/4$ cycles $(1_i, 2_j, 3_i, 4_j, \dots, (n-1)_i, n_j)$, $1 \leq i, j \leq m/2$. These new cycles can be partitioned into $m/2$ resolution classes, by using a quasi-group (Q, \circ) of order $m/2$. Collect together all cycles $(1_i, 2_j, 3_i, 4_j, \dots, (n-1)_i, n_j)$ with $i \circ j = \ell$ for each element $\ell \in Q$; these form one resolution class, for each ℓ . \square

Example 2.3. A gregarious resolvable 6-cycle decomposition of $K_{6(4)}$ from one of $K_{6(2)}$.

When $n = 6$, Lemma 2.1 yields the starter cycles (mod 5):

$$(\infty, 3, 0, 1', 4, 2'), \quad (\infty', 2, 1, 0', 3', 4').$$

Note that these do indeed form a resolution class. Then replacing parts $\{i, i'\}$ by $\{i_1, i_2, i'_1, i'_2\}$ we obtain the starter cycles (in two resolution classes)

$$\begin{aligned} &(\infty_1, 3_1, 0_1, 1'_1, 4_1, 2'_1), \quad (\infty'_1, 2_1, 1_1, 0'_1, 3'_1, 4'_1), (\infty_2, 3_2, 0_2, 1'_2, 4_2, 2'_2), \quad (\infty'_2, 2_2, 1_2, 0'_2, 3'_2, 4'_2); \\ &(\infty_1, 3_2, 0_1, 1'_2, 4_1, 2'_2), \quad (\infty'_1, 2_2, 1_1, 0'_2, 3'_1, 4'_2), (\infty_2, 3_1, 0_2, 1'_1, 4_2, 2'_1), \quad (\infty'_2, 2_1, 1_2, 0'_1, 3'_2, 4'_1). \end{aligned}$$

Cycled modulo 5, these yield 10 resolution classes in total, as required for $K_{6(4)}$. \square

3. Results: number of parts n is odd

Lemma 3.1. *There is a gregarious n -cycle decomposition of $K_{n(m)}$ when n is odd.*

Proof. Certainly K_n has a hamilton cycle decomposition whenever n is odd. Suppose any one cycle is $(1, 2, \dots, n)$. Now replace each vertex x in $V(K_n)$ by the set $\{x_i \mid 1 \leq i \leq m\}$. Let $\{\{1, 2, \dots, m\}, \circ\}$ be any quasigroup of order m , and replace the cycle $(1, 2, \dots, n)$ by the m^2 cycles

$$(1_i, 2_j, 3_i, \dots, (n-1)_j, n_k), \quad 1 \leq i, j \leq m, \quad k = i \circ j.$$

This gives a gregarious n -cycle decomposition of $K_{n(m)}$ when n is odd. \square

Using two orthogonal latin squares or quasigroups of order m , we can obtain a *resolvable* gregarious n -cycle decomposition of $K_{n(m)}$ with n odd, provided, of course, that $m \neq 2, 6$. So we have the following.

Corollary 3.2. *There is a resolvable gregarious n -cycle system of $K_{n(m)}$ with n odd, whenever $m \neq 2, 6$.*

Proof. Let (Q, \circ) and $(Q, *)$ be two orthogonal quasigroups of order m , with $Q = \{1, 2, \dots, m\}$, where necessarily $m \neq 2, 6$. From each cycle in a hamilton decomposition of K_n , we form m^2 cycles using (Q, \circ) as described in Lemma 3.1. Then for each $\ell \in Q$, and for each hamilton cycle, we take all new cycles arising, using those pairs i, j with $i * j = \ell$ in the second quasigroup $(Q, *)$; these form a resolution class. \square

Example 3.3. A resolvable gregarious 5-cycle decomposition of $K_{5(3)}$.

Take a hamilton decomposition of K_5 , with $V(K_5) = \{1, 2, 3, 4, 5\}$, to be $(1, 2, 3, 4, 5)$, $(1, 3, 5, 2, 4)$, and let the parts of size 3 be $\{i_1, i_2, i_3\}$ for $1 \leq i \leq 5$.

We use the following two orthogonal quasigroups of order 3.

\circ	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

$*$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

The first quasigroup yields 18 5-cycles:

$$\begin{aligned} &[1 \circ 1 = 1], \quad (1_1, 2_1, 3_1, 4_1, 5_1)^a, \quad (1_1, 3_1, 5_1, 2_1, 4_1)^A, \\ &[1 \circ 2 = 2], \quad (1_1, 2_2, 3_1, 4_2, 5_2), \quad (1_1, 3_2, 5_1, 2_2, 4_2), \\ &[1 \circ 3 = 3], \quad (1_1, 2_3, 3_1, 4_3, 5_3), \quad (1_1, 3_3, 5_1, 2_3, 4_3), \\ &[2 \circ 1 = 3], \quad (1_2, 2_1, 3_2, 4_1, 5_3), \quad (1_2, 3_1, 5_2, 2_1, 4_3), \\ &[2 \circ 2 = 1], \quad (1_2, 2_2, 3_2, 4_2, 5_1), \quad (1_2, 3_2, 5_2, 2_2, 4_1), \end{aligned}$$

$$\begin{aligned}
[2 \circ 3 = 2], & \quad (1_2, 2_3, 3_2, 4_3, 5_2)^a, \quad (1_2, 3_3, 5_2, 2_3, 4_2)^A, \\
[3 \circ 1 = 2], & \quad (1_3, 2_1, 3_3, 4_1, 5_2), \quad (1_3, 3_1, 5_3, 2_1, 4_2), \\
[3 \circ 2 = 3], & \quad (1_3, 2_2, 3_3, 4_2, 5_3)^a, \quad (1_3, 3_2, 5_3, 2_2, 4_3)^A, \\
[3 \circ 3 = 1], & \quad (1_3, 2_3, 3_3, 4_3, 5_1), \quad (1_3, 3_3, 5_3, 2_3, 4_1).
\end{aligned}$$

When $\ell = 1$, from the second quasigroup with operation $*$, we note that $1 * 1 = 1$, $2 * 3 = 1$ and $3 * 2 = 1$. So the cycles marked a and A above yield two of the six resolution classes. The reader may consider $\ell = 2, 3$ and the quasigroup with operation $*$ to find the remaining four resolution classes. \square

Note that the “obvious” decomposition of $K_{n(m)}$ into gregarious n -cycles given in Lemma 3.1 is not readily resolvable when $m = 2$ or 6 . So from now on we deal with the cases when the part size m is 2 , and $n \equiv 1 \pmod{4}$, $n \equiv 3 \pmod{4}$. Part size 6 will then follow from part size 2 , using two orthogonal quasigroups of order 3 .

3.1. Part size 2; number of parts $n \equiv 1 \pmod{4}$

Lemma 3.4. *There is a resolvable, gregarious n -cycle decomposition of $K_{n(2)}$ with $n \equiv 1 \pmod{4}$.*

Proof. Let $n = 4k + 1$.

Colour the edges of $2K_n$ red and blue so that each pair of vertices is joined by one red edge and one blue edge. Take a hamilton decomposition of K_n with the property that each hamilton cycle contains an *even* number of red edges. Here is one such hamilton decomposition:

Let $V(K_n) = \{\infty\} \cup \{0, 1, \dots, 4k - 1\}$, and take hamilton cycles

$$(\infty, 0, 1, 4k - 1, 2, 4k - 2, 3, \dots, 2k + 1, 2k) + i, \quad 0 \leq i \leq 2k - 1,$$

where all edges except $\{0, 1\}, \{1, 2\}, \{2, 3\}, \dots, \{2k - 1, 2k\}$ are red. Here, of course, the addition of i means that i is added to each entry (other than ∞) in the hamilton cycle, with addition modulo $4k$. So each cycle here contains one blue edge, and thus $4k$ (even) red edges. Note that the total number of red edges (which must be even) is $n(n - 1)/2$, so n , which is odd, must be $1 \pmod{4}$.

The remaining copy of K_n then has these $(n - 1)/2 = 2k$ edges red and the rest blue. We take one hamilton cycle containing all these $2k$ red edges:

$$H = (\infty, 0, 1, 2, \dots, 2k - 1, 2k, \dots, 4k - 1).$$

The rest, $K_n - H$, is known to be decomposable into a further $2k - 1$ hamilton cycles.

We now take two copies of these $4k$ hamilton cycles which exactly cover $2K_n$, and replace each red edge $\{x, y\}$ in one cycle by $\{x, y'\}$ and in the second cycle by $\{x', y\}$. Each blue edge $\{x, y\}$ is then $\{x, y\}$ in one cycle and $\{x', y'\}$ in the other, since there is an even number of red edges in each cycle.

The resulting $8k$ cycles form a resolvable gregarious n -cycle system of $K_{n(2)}$.

This completes the proof. \square

Corollary 3.5. *There is a resolvable, gregarious n -cycle decomposition of $K_{n(6)}$ with $n \equiv 1 \pmod{4}$.*

Proof. We take two orthogonal quasigroups of order 3 , on the set $\{1, 2, 3\}$, with operations \circ and $*$. Using the first quasigroup, we obtain a gregarious n -cycle system of $K_{n(6)}$, by replacing each cycle of the form $(1, 2, 3, 4, \dots, n)$ in the gregarious resolvable decomposition of $K_{n(2)}$ by the nine cycles $(1_i, 2_j, 3_i, 4_j, \dots, (n - 1)_j, n_k)$ where $k = i \circ j$, for all nine pairs i, j with $1 \leq i, j \leq 3$.

Then to obtain the resolution classes, from each old resolution class of size 2 in the gregarious resolvable decomposition of $K_{n(2)}$, we obtain three new resolution classes of size 6 as follows, making a total of $3(n - 1)$ resolution classes

containing the $18(n-1)$ cycles. For $\ell = 1, 2, 3$ in turn, take all pairs i, j with $i * j = \ell$ (there will be three such) and collect together those n -cycles $(1_i, 2_j, 3_i, 4_j, \dots, (n-1)_j, n_k)$ where $k = i \circ j$ and $\ell = i * j$.

The result is a resolvable and gregarious n -cycle system of $K_{n(6)}$. \square

Example 3.6. A resolvable gregarious 5-cycle decomposition of $K_{5(6)}$.

We apply the above lemma and corollary to K_5 . With vertex set $\{\infty, 0, 1, 2, 3\}$ and colouring each edge both red and blue, we obtain hamilton cycles $(\infty, 0, 1, 3, 2)$, $(\infty, 1, 2, 0, 3)$ where edges $\{0, 1\}$ and $\{1, 2\}$ are blue and the other eight are red. Then (with the second copy of K_5 and the remaining coloured edges) we take cycles $(\infty, 0, 1, 2, 3)$ and $(\infty, 1, 3, 0, 2)$ where the edges $\{0, 1\}$ and $\{1, 2\}$ are red and the rest are blue.

These yield the following four parallel classes of gregarious 5-cycles:

$$\begin{aligned} &(\infty, 0', 1', 3, 2'), & (\infty', 0, 1, 3', 2); \\ &(\infty, 1', 2', 0, 3'), & (\infty', 1, 2, 0', 3); \\ &(\infty, 0, 1', 2, 3), & (\infty', 0', 1, 2', 3'); \\ &(\infty, 1, 3, 0, 2), & (\infty', 1', 3', 0', 2'). \end{aligned}$$

Next, as described in the above corollary, we use the same orthogonal quasigroups of order 3 as given in Example 3.3 above.

Each partite set $\{i, i'\}$ of $K_{n(2)}$ now becomes the partite set $\{i_1, i_2, i_3, i'_1, i'_2, i'_3\}$ of size 6.

The first resolution class above yields eighteen 5-cycles, nine from each cycle:

$$\begin{aligned} [1 \circ j = k], & \quad (\infty_1, 0'_1, 1'_1, 3_1, 2'_1)^a, & (\infty_1, 0'_2, 1'_1, 3_2, 2'_2), & (\infty_1, 0'_3, 1'_1, 3_3, 2'_3), \\ [2 \circ j = k], & \quad (\infty_2, 0'_1, 1'_2, 3_1, 2'_3), & (\infty_2, 0'_2, 1'_2, 3_2, 2'_1), & (\infty_2, 0'_3, 1'_2, 3_3, 2'_2)^a, \\ [3 \circ j = k], & \quad (\infty_3, 0'_1, 1'_3, 3_1, 2'_2), & (\infty_3, 0'_2, 1'_3, 3_2, 2'_3)^a, & (\infty_3, 0'_3, 1'_3, 3_3, 2'_1); \\ \\ [1 \circ j = k], & \quad (\infty'_1, 0_1, 1_1, 3'_1, 2_1)^a, & (\infty'_1, 0_2, 1_1, 3'_2, 2_2), & (\infty'_1, 0_3, 1_1, 3'_3, 2_3), \\ [2 \circ j = k], & \quad (\infty'_2, 0_1, 1_2, 3'_1, 2_3), & (\infty'_2, 0_2, 1_2, 3'_2, 2_1), & (\infty'_2, 0_3, 1_2, 3'_3, 2_2)^a, \\ [3 \circ j = k], & \quad (\infty'_3, 0_1, 1_3, 3'_1, 2_2), & (\infty'_3, 0_2, 1_3, 3'_2, 2_3)^a, & (\infty'_3, 0_3, 1_3, 3'_3, 2_1). \end{aligned}$$

There are three other sets of 18 cycles like this, from the remaining three resolution classes in the decomposition of $K_{5(2)}$. The above 18 cycles form three parallel classes of size 6: when $\ell = 1$ in the quasigroup with operation $*$, we have pairs $(1, 1)$, $(2, 3)$, $(3, 2)$, and we take the six cycles marked with a above. Similarly when $\ell = 2, 3$, we pick six of the above each time, for those pairs i, j with $i * j = \ell$. This yields a resolvable gregarious 5-cycle decomposition of $K_{5(6)}$. \square

3.2. Part size 2; number of parts $n \equiv 3 \pmod{4}$

Here we deal with the case of a resolvable, gregarious n -cycle decomposition of $K_{n(2)}$ when $n \equiv 3 \pmod{4}$. In the case $n = 3$ we have an exact equivalence with a pair of orthogonal quasigroups of order m , so we know there is no resolvable 3-cycle decomposition of $K_{3(2)} = K_{2,2,2}$ or of $K_{3(6)}$. However, a resolvable gregarious 7-cycle decomposition of $K_{7(2)}$ was found by Dukes [3]; see the following example.

Example 3.7 (Dukes [3]). A gregarious, resolvable 7-cycle system of $K_{7(2)}$.

Let the vertex set of $K_{7(2)}$ be $\{i, i' \mid 1 \leq i \leq 7\}$. Then resolution classes are (the six rows):

$$\begin{aligned} &(0, 1, 3', 2, 4, 5', 6'), & (0', 5, 1', 4', 2', 6, 3); \\ &(0, 1', 4, 6, 5, 2', 3), & (0', 1, 4', 6', 2, 5', 3'); \\ &(0', 1', 2', 3', 5, 4, 6'), & (0, 2, 6, 4', 3, 1, 5'); \\ &(0, 2', 6', 5, 1, 4, 3'), & (0', 5', 4', 2, 3, 1', 6); \\ &(0', 2', 1, 6, 5', 3, 4), & (0, 5, 2, 1', 6', 3', 4'); \\ &(0', 2, 1, 6', 3, 5, 4'), & (0, 6, 3', 1', 5', 2', 4). \end{aligned}$$

Lemma 3.8. Let $k \geq 2$. There exists a hamilton decomposition $\{H(i) \mid i \in \mathbb{Z}_{2k+1}\}$ of K_{4k+3} for which there exists a set of edges $\{e(i) \mid e(i) \in E(H(i)), i \in \mathbb{Z}_{2k+1}\}$ satisfying:

- (1) $E_1 = \{e(i) \mid i \in \mathbb{Z}_5\}$ is an independent set, and
- (2) each edge in $E_2 = \{e(i) \mid 5 \leq i \leq 2k\}$ is incident with a vertex α , and
- (3) no vertex is incident with an edge in E_1 and an edge in E_2 .

Proof. The classic hamilton decomposition of K_{4k+3} on the vertex set $\{\infty\} \cup \mathbb{Z}_{4k+2}$ is given by defining $H(i) = (v_{i,0}, v_{i,1}, \dots, v_{i,4k+2})$ where

$$v_{i,j} = \begin{cases} (-1)^j \lceil j/2 \rceil + i + 1 & \text{if } j \in \mathbb{Z}_{4s+2}, \\ \infty & \text{if } j = 4k + 2 \end{cases}$$

for each $i \in \mathbb{Z}_{2k}$. Then since $k \geq 2$, the result can be obtained by defining $e(0) = \{0, 1\}$, $e(1) = \{2k + 2, 2k + 3\}$, $e(2) = \{2, 3\}$, $e(3) = \{2k + 4, 2k + 5\}$, $e(4) = \{4, 5\}$, and $e(i) = \{\infty, i + 1\}$ for $5 \leq i \leq 2k$ (so $\alpha = \infty$). \square

We will need the two following lemmas. They are tantalizingly close to results that follow from [4], but the first needs setting up differently.

An edge-colouring of a multigraph is said to be balanced if the edges are shared out as evenly as possible among the edges at each vertex, and are shared out as evenly as possible among the edges between each pair of vertices. It has been proved that for each positive integer y and for each bipartite multigraph G there exists a balanced y -edge-colouring of G [7]. Let G_y denote the subgraph of G induced by the edges coloured y , and let ε_y denote the number of edges coloured y .

Lemma 3.9. Let $k \geq 3$. Let T' be a copy of K_{10} on the vertex set \mathbb{Z}_{10} with edges coloured using colours in \mathbb{Z}_{2k+2} such that:

1. the edges coloured 0 are precisely those in $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\} \cup \{\{1, 2\}, \{5, 6\}\} \cup \{\{7, 5\}, \{5, 8\}\}$;
2. the edge $\{0, 3\}$ is coloured 2; and
3. the remaining edges are coloured with colours $3, \dots, 2k + 1$ so that for $3 \leq i \leq 2k + 1$, colour class i :
 - (a) has maximum degree at most 2;
 - (b) contains no cycles; and
 - (c) has at least $17 - 4k$ edges.

Then this edge-coloured graph $G = K_{10}$ can be embedded in an edge-coloured K_{4k+3} on the vertex set \mathbb{Z}_{4k+3} in which:

- (a) the edges outside G are coloured with colours $1, \dots, 2k + 1$;
- (b) colour class 1 is a path of length $4k - 2$ from vertex 0 to vertex 3 that avoids the vertices in $\{1, 2, 5, 6\}$;
- (c) colour class 2 is a path of length $4k - 1$ from vertex 4 to vertex 9 that avoids the vertices in $\{5, 7, 8\}$; and
- (d) for $3 \leq i \leq 2n + 1$, colour class i induces a hamilton cycle.

Remark. It is easy to obtain such an edge-colouring of K_{10} (when $k \geq 5$ a proper edge-colouring will suffice). Also, note that the edges coloured 0, 1 and 2 together induce a 4-regular subgraph of K_{4k+3} .

Proof. Form a multigraph $G(0)$ from G by adding one vertex α , then join α to:

1. vertices 0 and 3 with one edge and vertices 4, 7, 8, and 9 with two edges, each of which is coloured 1;
2. vertices 0, 3, 4, and 9 with one edge and vertices 1, 2, and 6 with two edges, each of which is coloured 2; and
3. for $3 \leq i \leq 2k + 1$, and for each $j \in \mathbb{Z}_{10}$, join vertex j to α with $2 - d_{G(0)}(j)$ edges coloured i .

Each vertex v in G is joined to nine other vertices, so by considering each of the $2k + 1$ colours, one can check that v is joined to α in $G(0)$ with $2(2k + 1) - 9 = 4k - 7$ edges. Also, by condition (3(c)), for $3 \leq i \leq 2k + 1$, α is incident

with exactly $20 - 2\varepsilon_i \leq 20 - 2(17 - 4k) = 8k - 14 = 2(4k - 7)$ edges coloured i ; and for $1 \leq i \leq 2$, v is incident with exactly $10 \leq 2(4k - 7)$ (since $k \geq 3$) edges coloured i . Therefore, to complete the formation of $G(0)$, for $3 \leq i \leq 2k + 1$ add to α $(8k - 14 - (20 - 2\varepsilon_i))/2 \geq 0$ loops coloured i , and for $1 \leq i \leq 2$ add to α $(8k - 14 - 10)/2 \geq 0$ loops coloured i .

Then $d_{G(0)_i}(\alpha) = 8k - 14$ for $1 \leq i \leq 2k + 1$. Also, the number of loops on α is $((2k + 1)(8k - 14) - (4k - 7)10)/2 = (4k - 7)(2k - 4)$, the number of edges in K_{4k-7} .

The proof now follows closely the proof of Theorem 3.1 in [4]. The one difficulty to attend to here is to make sure that the colour classes 1 and 2 are indeed connected paths (with no cycles broken off) by the time we are finished. Because of the similarity, the following is brief and to the point.

The technique is to produce a family of graphs $G(0), G(1), \dots, G(4k - 8)$ in which for $0 \leq x \leq 4k - 8$, $G(x)$ satisfies the following properties:

- (P_1) the vertex set is $\mathbb{Z}_{10+x} \cup \{\alpha\}$;
- (P_2) the number of edges between each pair of vertices is 1, unless one of the vertices is α in which case the number is $4k - 7 - x$;
- (P_3) the number of loops on α is $(4k - 7 - x)(4k - 8 - x)/2$, and there are no other loops; and
- (P_4) for $1 \leq i \leq 2k + 1$, colour class i is connected, each vertex other than α is incident with at most two edges of each colour, and α has degree $2(4k - 7 - x)$ in each colour class (loops contribute 2 to the degree of the vertex).

Clearly $G(0)$ has been shown to satisfy these properties. Furthermore, if $G(4k - 8)$ exists, then after renaming α with $4k + 2$, it is the graph we are seeking: by (P_2) and (P_3) it is a complete graph, and it satisfies properties ($b - d$) because of (P_4).

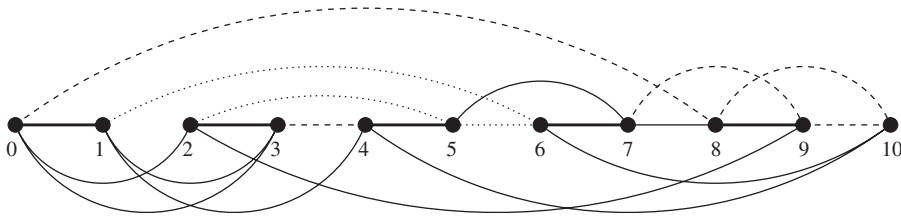
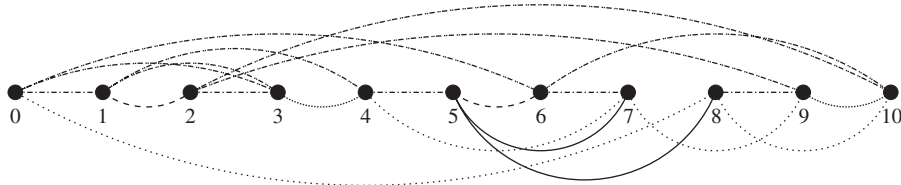
The construction of this sequence of graphs proceeds inductively. So suppose that $G(x)$ exists, for some $x \geq 1$. Let B be a bipartite graph with bipartition $\{c_1, \dots, c_{2k+1}\}$ and $\mathbb{Z}_{10+x} \cup \{l\}$ of the vertex set. For $1 \leq i \leq 2k + 1$ and each $z \in \mathbb{Z}_{10+x}$, join vertices c_i and z in B with the number of edges coloured i joining vertex z to α in $G(x)$, and join c_i to l with twice the number of loops coloured i on α . Then in B , c_i has degree $2(4k - 7 - x)$ by (P_4), z has degree $4k - 7 - x$ by (P_2), and l has degree $(4k - 7 - x)(4k - 8 - x)$ by (P_3). Give B a balanced edge-colouring with $(4k - 7 - x)$ colours, and let B_1 be the subgraph of B induced by any two colour classes. So in B_1 , c_i has degree 4, z has degree 2, and l has degree $2(4k - 8 - x)$.

To attend to the connectivity issue in (P_4), form the bipartite graph B_2 from B_1 by adding a new vertex c'_i for $1 \leq i \leq 2k + 1$, then detaching two of the four edges incident with c_i and joining them to c'_i instead, the two detached edges being chosen as follows.

1. If the two edges coloured 1 in $G(0)$ that join α to 1 and to 3 correspond to edges in B_1 then the corresponding edges are adjacent in B_2 .
2. If the two edges coloured 2 in $G(0)$ that join α to 4 and to 9 correspond to edges in B_1 then the corresponding edges are adjacent in B_2 .
3. For $1 \leq i \leq 2k + 1$, and for each component of $G(x - 1)_i - \alpha$ that is joined to α by exactly two edges, if these two edges correspond to edges in B_1 then the corresponding edges are adjacent in B_2 .
4. If l is joined to c_i with at least two edges in B_1 then c_i is joined to l with exactly two edges in B_2 .

Then in B_2 , c_i, c'_i have degree 2, z has degree 2, and l has degree $2(4k - 8 - x)$.

Give B_2 a balanced edge-colouring with two colours, say a and b . To form $G(x + 1)$, add a new vertex $10 + x$ to $G(x)$ so the vertex set of $G(x)$ is $\mathbb{Z}_{10+(x+1)} \cup \{\alpha\}$. For each edge in B_2 coloured a that joins c_i or c'_i to z , detach an edge coloured i joining z to α in $G(x)$ from α and join it to the new vertex $10 + x$ instead; and for each edge in B_2 coloured a that joins c_i or c'_i to l , detach one end of a loop coloured y on α in $G(x)$ and join it to the new vertex $10 + x$ instead (so the loop becomes an edge joining z and α in $G(x)$). Then the new vertex: is incident with exactly two edges of each colour i (since each of c_i and c'_i is incident with one edge coloured a in B_2); is joined to each vertex $z \in \mathbb{Z}_{10+x}$ with one edge (since z is incident with one edge coloured a in B_2); and is joined to α by exactly $4x + 8 - x = 4x + 7 - (x + 1)$ edges (since l is incident with $(4k - 8 - x)$ edges coloured a in B_2). The manoeuvre performed in constructing B_2 from B_1 ensures that each colour class is connected. It is not hard to check that the remaining properties required are satisfied by this graph, so $G(x + 1)$ has been formed as required. \square

Fig. 1. A possible S when $k = 2$.Fig. 2. A possible T when $k = 2$.

The following result can be proved in the same way as the previous result. (It also follows directly from Theorem 3.1 in [4], since in this case the edges coloured 0, 1 and 2 could be partitioned into two 2-factors, each of which contains one of the long paths; this is not possible in the previous lemma.)

Lemma 3.10. *Let $k \geq 3$. Let S' be a copy of K_{10} on the vertex set \mathbb{Z}_{10} with edges coloured using colours in \mathbb{Z}_{2k+2} such that:*

1. *the edges coloured 0 are precisely those in $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\} \cup \{\{1, 6\}, \{6, 5\}, \{5, 2\}\} \cup \{\{3, 4\}\} \cup \{\{5, 7\}, \{7, 8\}\}$;*
2. *the edge $\{1, 2\}$ is coloured 2; and*
3. *the remaining edges are coloured with colours $3, \dots, 2k+1$ so that for $3 \leq i \leq 2k+1$, colour class i :*
 - (a) *has maximum degree at most 2;*
 - (b) *contains no cycles; and*
 - (c) *has at least $17 - 4k$ edges.*

Then this edge-coloured graph $G = K_{10}$ can be embedded in an edge-coloured K_{4k+3} on the vertex set \mathbb{Z}_{4k+3} in which:

- (a) *the edges outside G are coloured with colours $1, \dots, 2k+1$;*
- (b) *colour class 1 is a path of length $4k-4$ from vertex 0 to vertex 7 that avoids the vertices in $\{1, \dots, 6\}$;*
- (c) *colour class 2 is a path of length $4k-1$ from vertex 6 to vertex 9 that avoids the vertices in $\{5, 7, 8\}$, and*
- (d) *for $3 \leq i \leq 2n+1$, colour class i induces a hamilton cycle.*

Remark. It is easy to obtain such an edge-colouring of K_{10} (when $k \geq 5$ a proper edge-colouring will suffice).

Finally, we need two particular decompositions of K_{11} . Each of the following is a set of edge-disjoint hamilton cycles, the first being the complement in K_{11} of the graph S in Fig. 1, the second being the complement in K_{11} of the graph T in Fig. 2: $\{(0, 4, 7, 10, 3, 6, 2, 1, 8, 5, 9), (0, 5, 1, 10, 2, 4, 8, 6, 9, 3, 7), (0, 6, 4, 9, 1, 7, 2, 8, 3, 5, 10)\}$; $\{(0, 2, 8, 3, 9, 6, 4, 10, 7, 1, 5), (0, 4, 2, 6, 1, 8, 7, 3, 10, 5, 9), (0, 7, 2, 5, 3, 6, 8, 4, 9, 1, 10)\}$.

Theorem 3.11. *There exists a resolvable gregarious $(4k+3)$ -cycle system of $K_{(4k+3)(2)}$ whenever $k \geq 2$.*

Proof. By Lemma 3.8, there exists a hamilton decomposition $\{H(i) \mid i \in \mathbb{Z}_{2k+1}\}$ of K_{4k+3} on the vertex set \mathbb{Z}_{4k+3} in which:

- (1) $H(i)$ contains the edge $e(i) = \{2i, 2i + 1\}$ for $i \in \mathbb{Z}_5$; and
- (2) for each $i \in \mathbb{Z}_{2k+1} \setminus \mathbb{Z}_5$, $H(i)$ contains the edge $e(i) = \{4k + 2, 4k + 6 - i\}$.

For each $i \in \mathbb{Z}_{4k+3}$ let $H_1(i)$ be the graph defined on $\mathbb{Z}_{4k+3} \times \mathbb{Z}_2$ induced by the edge set

$$\{(u, 0), (v, 1)\}, \{(u, 1), (v, 0)\} \mid \{(u, v)\} \in E(H(i)) \setminus \{e(i)\}\} \cup \{(u, 0), (v, 0)\}, \{(u, 1), (v, 1)\} \mid \{u, v\} = e(i)\}.$$

Then each component of $H_1(i)$ is a gregarious $4k + 3$ -cycle, and $H_1(i)$ spans $K_{(4k+3)(2)}$.

Let S be the 4-regular graph on the vertex set \mathbb{Z}_{4k+3} induced by the union of the following five disjoint sets of edges (S corresponds to the subgraph of K_{4k+3} induced by the edges coloured 0, 1 and 2 in Lemma 3.10):

- (S1) $\{e(i) \mid i \in \mathbb{Z}_5\}$.
- (S2) The edges in the path $P_2 = (1, 6, 5, 2)$.
- (S3) The edge $\{3, 4\}$ together with the edges in a path P_3 of length $4k - 4$ from vertex 0 to vertex 7 that avoids both the vertices in $\{1, 2, \dots, 6\}$ and the edges in $\{e(i) \mid i \in \mathbb{Z}_{4k+3} \setminus \mathbb{Z}_5\}$.
- (S4) The edges in the path $P_4 = (5, 7, 8)$.
- (S5) The edges in a path P_5 of length $4k - 1$ from vertex 6 to vertex 9 that avoids both the vertices in $\{5, 7, 8\}$ and the edges in $\{e(i) \mid i \in \mathbb{Z}_{4k+3} \setminus \mathbb{Z}_5\}$.

(See Fig. 1.)

Let T be the 4-regular graph on the vertex set \mathbb{Z}_{4k+3} induced by the union of the following five sets of edges (T corresponds to the subgraph of K_{4k+3} induced by the edges coloured 0, 1 and 2 in Lemma 3.9).

- (T1) $\{e(i) \mid i \in \mathbb{Z}_5\}$.
- (T2) The edges in a path Q_2 of length $4k - 2$ from vertex 0 to vertex 3 that avoids both the vertices in $\{1, 2, 5, 6\}$ and the edges in $\{e(i) \mid i \in \mathbb{Z}_{4k+3} \setminus \mathbb{Z}_5\}$.
- (T3) The edges in $Q_3 = \{\{1, 2\}, \{5, 6\}\}$.
- (T4) The edges in a path Q_4 of length $4k - 1$ from vertex 4 to vertex 9 that avoids both the vertices in $\{5, 7, 8\}$ and the edges in $\{e(i) \mid i \in \mathbb{Z}_{4k+3} \setminus \mathbb{Z}_5\}$.
- (T5) The edges in the path $Q_5 = (7, 5, 8)$.

(See Fig. 2.)

Let S_1 be formed from S by renaming vertex i with $(i, 0)$ for each $i \in \mathbb{Z}_{4k+3}$, and form T_1 from T by renaming i with $(i, 1)$.

Each of the four following sets of edges induces a 2-regular subgraph of $K_{(4k+3)(2)}$ on the vertex set $\mathbb{Z}_{4k+3} \times \mathbb{Z}_2$ in which each component is a gregarious $(4k + 3)$ -cycle:

1. The edges in S_1 and T_1 corresponding to those defined in (S2) and (T2), together with the edges in $\{(1, 0), (0, 1)\}, \{(2, 0), (3, 1)\}$.
2. The edges in S_1 and T_1 corresponding to those defined in (S3) and (T3), together with the edges in $\{(0, 0), (1, 1)\}, \{(3, 0), (2, 1)\}, \{(4, 0), (5, 1)\}, \{(7, 0), (6, 1)\}$.
3. The edges in S_1 and T_1 corresponding to those defined in (S4) and (T4), together with the edges in $\{(5, 0), (4, 1)\}, \{(8, 0), (9, 1)\}$.
4. The edges in S_1 and T_1 corresponding to those defined in (S5) and (T5), together with the edges in $\{(6, 0), (7, 1)\}, \{(9, 0), (8, 1)\}$.

Finally we consider the remaining edges, namely:

- (a) edges joining vertices $\{(u, 0), (v, 0)\}$ where $\{u, v\} \notin E(S)$;

- (b) edges joining vertices $\{(u, 1), (v, 1)\}$ where $\{u, v\} \notin E(T)$; and
- (c) edges in $\{ \{(u, 0), (v, 1)\}, \{(u, 1), (v, 0)\} \mid \{u, v\} = e(i), 5 \leq i \leq 2k \}$.

When $k \geq 3$ then by Lemma 3.10, and when $k = 2$ then by the hamilton decomposition of $K_{11} - E(S)$ that follows Lemma 3.10, there exists a hamilton decomposition $\{G(i) \mid i \in \mathbb{Z}_{2k-1}\}$ of $K_{4k+3} - E(S_1)$ on the vertex set $\mathbb{Z}_{4k+3} \times \{0\}$; we can assume that $G(i)$ contains the edges $\{(4k+2, 0), (4k+1-2i, 0)\}$ and $\{(4k+2, 0), (4k-2i, 0)\}$ for $i \in \mathbb{Z}_{k-2}$ (these edges correspond to the edges $e(j)$ for $j \in \mathbb{Z}_{2k+1} \setminus \mathbb{Z}_5$, and already occur in $H_1(j)$). Form $G_1(i)$ from $G(i)$ as follows: for each $i \in \mathbb{Z}_{k-2}$ replace the two edges incident with $(4k+2, 0)$ with $\{(4k+1-2i, 0), (4k+2, 1)\}$ and $\{(4k-2i, 0), (4k+2, 1)\}$, and otherwise let $G_1(i) = G(i)$.

Similarly, when $k \geq 3$ then by Lemma 3.9, and when $k = 2$ then by the hamilton decomposition of $K_{11} - E(T)$ that follows Lemma 3.10, there exists a hamilton decomposition $\{J(i) \mid i \in \mathbb{Z}_{2k-1}\}$ of $K_{4k+3} - E(T_1)$ on the vertex set $\mathbb{Z}_{4k+3} \times \{1\}$; we can assume that $J(i)$ contains the edges $\{(4k+2, 1), (4k+1-2i, 1)\}$ and $\{(4k-2i, 1), (4k+2, 1)\}$ for $i \in \mathbb{Z}_{k-2}$. Form $J_1(i)$ from $J(i)$ as follows: for each $i \in \mathbb{Z}_{k-2}$ replace the two edges incident with $(4k+2, 1)$ with $\{(4k+1-2i, 1), (4k+2, 0)\}$ and $\{(4k-2i, 1), (4k+2, 0)\}$, and otherwise let $G_1(i) = G(i)$.

Then for each $i \in \mathbb{Z}_{2k-1}$, $G_1(i) \cup J_1(i)$ is a 2-regular spanning subgraph of $K_{(4k+3)(2)}$, each component of which is a gregarious $(4k+3)$ -cycle.

So the result is proved. \square

We summarise our results as follows.

Theorem 3.12. *There exists a resolvable edge-disjoint decomposition of a complete multipartite graph with n parts of equal size m into gregarious n -cycles for all m and n except when m is odd and n is even, or when $n = 3$ and $m = 2$ or 6; in these cases such a resolvable gregarious decomposition is impossible.*

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